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On strong convergence to 3-D axisymmetric vortex sheets

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Abstract

We consider the 3-D axisymmetric incompressible Euler equations without swirls with vortex-sheets initial data. It is proved that the approximate solutions, generated by smoothing the initial data, converge strongly in $L^2\left([0,T]; L^2_{\mathrm{loc}}(R^3)\right)$ provided that they have strong convergence in the region away from the symmetry axis. This implies that if there would appear singularity or energy lost in the process of limit for the approximate solutions, it then must happen in the region away from the symmetry axis. There is no restriction on the signs of initial vorticity here. In order to exclude the possible concentrations on the symmetry axis, we use the special structure of the equations for axisymmetric flows and careful choice of test functions.

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1. Introduction

Consider the 3-D incompressible Euler equations in R^3

$$\begin{aligned} &\partial_t u + (u \cdot \nabla)u + \nabla p = 0, \quad (x,t) \in R^3 \times (0,T), \\ &\operatorname{div} u = 0, \quad |u| \to 0 \quad \text{as} \quad |x| \to +\infty, \end{aligned} \tag{1.1}$$

with initial conditions

$$u(x,t)\,|_{t=0} = u_0(x). \tag{1.2}$$

Eqs. (1.1) describe motions of incompressible homogeneous inviscid flows. The unknown functions here are the velocity vector $u = (u_1(x,t), u_2(x,t), u_3(x,t))$ and the pressure $p(x,t)$. T is a fixed positive constant.

As is well known, the global solvability of system (1.1)–(1.2) is still an outstanding open problem in the mathematical fluid mechanics. The local existence and uniqueness of the classical solutions and the various criteria for development of singularities were shown in [1,10] for $\Omega = R^3$ and bounded domain, respectively. Even for the axisymmetric case, there is still no global solvability for (1.1)–(1.2), except for investigations of the possible development of singularities (see [3] and the references therein). If the axisymmetric flows have no swirls (the angular component of the velocity in the cylindrical coordinate system is zero), the global existence and uniqueness of regular solutions have been proved by Majda [16] and Saint-Raymond [18], respectively, and the global existence of weak solutions have also been studied (see [2] and the references therein).

In this paper, we are concerned with the strong convergence of the approximate solutions of the 3-D axisymmetric Euler equations without swirls when the initial data is a vortex-sheets data. The weak solution approach to the problem of vortex-sheets motions was initiated by the works of Diperna–Majada in [6–8]. It is known that for the 2-D Euler equations, if the initial data is a vortex-sheets, that is, if the initial vorticity $\omega_0(x) = curl\, u_0(x)$ satisfying

$$\omega_0 \in M(R^2) \cap H^{-1}_{\text{comp}}(R^2), \tag{1.3}$$

where $M(R^2)$ is the space of finite Radon measures and $H^{-1}_{\text{comp}}(R^2)$ is the dual space of usual Hilbert space $H^1(R^2)$ with compact support, Delort obtained the first existence of a global classical weak solution with the additional assumption that the initial vorticity ω_0 is of one-sign (see [4]). Furthermore, for the one-signed initial vorticity, the convergence to classical weak solutions of the vortex-sheets problem for various types of the approximate solutions has been established (see [17,13,14,9,19]). In the case that the vorticity may change signs, Lopes Filho–Nussenzveig Lopes–Xin established in [15] the global existence of a classical weak solution to the 2-D vortex-sheets problem for initial vorticity with reflection symmetry. However, for the 3-D axisymmetric flows without swirls, Delort proved in [5] that, if the vortex-sheets initial vorticity has distinguished sign, the sequence of approximate solutions obtained by smoothing the initial data either converges strongly in $L^2_{\mathrm{loc}}(R^3 \times (0, +\infty))$ or converges weakly in $L^2_{\mathrm{loc}}(R^3 \times (0, +\infty))$ to a limit which is not a classical weak solution to the Euler equations in the sense of distribution, which implies that there is no concentration-cancellation occurring for the 3-D axisymmetric flows without swirls. This is in sharp contrast to the 2-D theory.

Our main result in this paper shows that for the approximate solutions generated by smoothing the initial data, if they converge strongly (in L^2-space) over the region outside the symmetry axis, then they must converge strongly in the whole space (see Theorem 3.2 in Section 3). This implies that for the 3-D axisymmetric Euler equations without swirls, if there would appear energy concentration in the process of limit for the approximate solutions, the set of energy-concentration must contain points in the region outside the symmetry axis. There is no restriction on the signs of the initial vorticity in our result.

Our analysis is based on some concentration–compactness arguments. To exclude the possible energy concentrations of the approximate solutions to the vortex-sheets problem on the symmetry axis, we make full use of the integral equations satisfied by the weak limits of $|u^\varepsilon|^2$ in the sense of measure and construct carefully various special test functions. One of the ingredients in our analysis is the uniform decay of the energy of the approximate solutions near the symmetry axis which is given by an estimate due to Chae and Imanuvilov [2] (see Proposition 2.3 in Section 2). In particular, in the case that the initial vorticity has one sign, our result can be obtained directly from Chae and Imanuvilov's result (Proposition 2.3 in Section 2) and Delort's result (Proposition 3.1 in Section 3).

Note that the structure of the 3-D axisymmetric Euler equations without swirls, (2.2)–(2.5) in Section 2, is very similar to the 2-D Euler system except at the symmetry axis $r = 0$. One could expect that the approximate solutions may become more singular near the symmetry axis. Our results in this paper imply that this is not the case, and the task to obtain the existence of classical weak solutions to the 3-D axisymmetric Euler equations without swirls is to show the strong convergence of the approximate solution sequence away from the symmetry axis.

The rest of this paper is organized as follows. In Section 2, we present the governing axisymmetric equations and some preliminary uniform estimates on the approximate solutions generated by smoothing the initial data. In Section 3, we state and prove the main result. In Section 4, we give some concluding remarks.

2. Axisymmetric equations and approximate solutions

The cylindrical transformation in R^3 is defined as

$$\begin{aligned}
&\pi : \bar{R}_+ \times [0, 2\pi) \times R \longrightarrow R^3,\\
&(r, \theta, z) \longmapsto (x_1, x_2, x_3),\\
&x_1 = r\cos\theta, \quad x_2 = r\sin\theta, \quad x_3 = z.
\end{aligned} \tag{2.1}$$

By axisymmetric solutions of (1.1), we mean that, in the cylindrical coordinate system, the unknown functions $u(x,t)$ and $p(x,t)$ do not depend on θ-variable, that is,

$$\begin{aligned} u(x,t) &= u^r(r,z,t)e_r + u^\theta(r,z,t)e_\theta + u^z(r,z,t)e_z, \\ p(x,t) &= p(r,z,t), \end{aligned}$$

where e_r, e_θ, e_z form the standard orthogonal bases in the cylindrical coordinate system. When $u^\theta \equiv 0$, the Euler equations (1.1) can be written as

$$\frac{\tilde{D}u^r}{Dt} + \partial_r p = 0, \tag{2.2}$$

$$\frac{\tilde{D}u^z}{Dt} + \partial_z p = 0. \tag{2.3}$$

$$\partial_r(ru^r) + \partial_z(ru^z) = 0. \tag{2.4}$$

In Eqs. (2.2)–(2.3) and in the following, we denote

$$\frac{\tilde{D}}{Dt} = \frac{\partial}{\partial t} + u^r\partial_r + u^z\partial_z, \quad r = \left(x_1^2 + x_2^2\right)^{1/2}.$$

Eqs. (2.2)–(2.4) are called 3-D axisymmetric Euler equations without swirls. In this case, the vorticity of the velocity is

$$\omega = \nabla \times u = \omega^\theta e^\theta,$$

with $\omega^\theta = \partial_z u^r - \partial_r u^z$ satisfying the following transport equation:

$$\frac{\tilde{D}}{Dt}\left(\frac{\omega^\theta}{r}\right) = 0. \tag{2.5}$$

For the 3-D axisymmetric Euler equations without swirls, the general vortex-sheets initial data can be described in the following way:

Assumptions (A). Suppose that $\omega_0^\theta = \omega_0^\theta(r,z)$ is a finite measure with compact support on $\left\{(r,z) \in R^2 \mid r \geqslant 0\right\}$, and that the initial vorticity $\omega_0(x)$, $x \in R^3$, defined by $\omega_0\,dx = \pi_*\left(r\omega_0^\theta(r,z)e_\theta\,d\theta\right)$, belongs to $H^{-1}(R^3)$, which guarantees the initial velocity $u_0(x) = -\triangle^{-1}\nabla \times \omega_0(x)$ belonging to $L^2(R^3)$.

Here π_* is the measure image induced by π which is defined in (2.1), i.e., assume that $d\mu$ is a measure on (r, θ, z)-space, then $\pi_* \, d\mu$ is a measure on (x_1, x_2, x_3)-space defined by

$$\int_{\Omega} g(x)(\pi_* \, d\mu) = \int_{\tilde{\Omega}} g \circ \pi \, d\mu, \tag{2.6}$$

where $\Omega = \pi(\tilde{\Omega})$ and $g(x) \in C_0(\Omega)$ is a continuous function with compact support in Ω.

For one-signed (non-negative, say) vortex-sheets initial data, we may assume that (see [5]).

Assumptions ($\mathbf{A}^+$). Assumptions (A) with $\omega_0^\theta \geqslant 0$ a positive measure.

The vortex-sheets initial data can be regularized through the usual way and the approximate solutions can be obtained by solving the 3-D axisymmetric Euler equations through smoothing the initial data. The detailed procedure can be found in [5] for one-signed case. We just state the results as follows:

Proposition 2.1. *Under Assumptions* (A), *the following statements hold*:

(1) *There exists a smooth sequence* $\left\{(\omega_0^\theta)^\varepsilon(r,z)\right\}$ *such that* $(\omega_0^\theta)^\varepsilon \in C_0^\infty((0,+\infty)\times R)$, *and*

$$\int_{-\infty}^{+\infty}\int_0^{+\infty} |(\omega_0^\theta)^\varepsilon| \, dr \, dz \leqslant C.$$

(2) *Let* $\omega_0^\varepsilon(x) = \left(\omega_0^\theta\right)^\varepsilon e_\theta$. *Then* $\omega_0^\varepsilon \in C_0^\infty\left(R^3 \setminus \left\{x \in R^3 \mid r = 0\right\}\right)$ *and* ω_0^ε *is uniformly bounded in* $H^{-1}(R^3) \cap L^1(R^3)$, *and*

$$\omega_0^\varepsilon \rightharpoonup \omega_0$$

weakly in $H^{-1}(R^3)$.

(3) *Let* $u_0^\varepsilon = -\nabla \times \Delta^{-1}\omega_0^\varepsilon$. *Then* $u_0^\varepsilon \in C^\infty(R^3)$, *and* u_0^ε *is uniformly bounded in* $L^2(R^3)$, *and* u_0^ε *is axisymmetric. Furthermore, we have* $(u_0^\varepsilon)^\theta \equiv 0$ *and*

$$u_0^\varepsilon \rightharpoonup u_0 = -\nabla \times \Delta^{-1}\omega_0$$

weakly in $L^2(R^3)$.

Proposition 2.2. *Under assumptions* (A), *there exist smooth approximate solutions* $u^\varepsilon, p^\varepsilon$ *of* 3-D *Euler equations* (1.1) *with initial data presented in Proposition* 2.1

such that for any $T > 0$,

(i)

$$u^{\varepsilon}(x,t) \text{ is uniformly bounded in } L^{\infty}([0,T]; L^2(R^3));$$

(ii) u^{ε} *is axisymmetric*, *i.e.*,

$$u^{\varepsilon}(x,t) = (u^r)^{\varepsilon} e_r + (u^z)^{\varepsilon} e_z;$$

(iii)

$$\omega^{\varepsilon} = \nabla \times u^{\varepsilon} = (\omega^{\theta})^{\varepsilon} e_{\theta}, \quad (\omega^{\theta})^{\varepsilon} = \partial_z (u^r)^{\varepsilon} - \partial_r (u^z)^{\varepsilon};$$

(iv) $(\omega^{\theta})^{\varepsilon}$ *is uniformly bounded in* $L^{\infty}(R; L^1(\bar{R_+} \times R, dr\, dz))$, *that is*

$$\max_{0 \leqslant t \leqslant T} \int_{-\infty}^{+\infty} \int_{0}^{+\infty} |(\omega^{\theta})^{\varepsilon}|\, dr\, dz \leqslant C;$$

where C *is a positive constant independent of* ε *and* T.

Remark 2.1. For one-signed vortex-sheets initial data stated as in Assumptions (A^+), the results of Propositions 2.1 and 2.2 still hold. Furthermore, in this case, one has the following additional estimate in (iv) of Proposition 2.2:

$$\max_{0 \leqslant t \leqslant T} \int_{-\infty}^{+\infty} \int_{0}^{+\infty} |(\omega^{\theta})^{\varepsilon}| r^2\, dr\, dz \leqslant C$$

with C a positive constant independent of ε and T, which results from the conservation of the fluid impulse $\frac{1}{2}\int_{R^3} x \times \omega\, dx$ (see [5]).

The following proposition, which is due to Chae and Imanuvilov (see [2] for details), will play a key role in our later analysis since it implies a uniform decay estimate of the energy for the approximations near the symmetry axis.

Proposition 2.3 (*Chae and Imanuvilov [2]*). *Let* $\{u^{\varepsilon}\}$ *and* $\{\omega^{\varepsilon}\}$ *be the approximate solutions constructed in Proposition* 2.2. *Then*, *for any* $T > 0$,

$$\int_0^T \int_{R^3} \frac{1}{1+x_3^2} \left(\frac{u_r^{\varepsilon}}{r}\right)^2 dx\, dt \leqslant C \left(\|u_0^{\varepsilon}\|_{L^2}^2 + \left\| \frac{\omega_0^{\varepsilon}}{r} \right\|_{L^1} \right), \tag{2.7}$$

where $C = C(T)$ *is a constant which does not depend on* ε.

Before we end this section, we recall the definition of the weak solutions to the 3-D Euler equations (1.1) in R^3 with initial data (1.2):

Definition 2.1. Suppose that $u_0(x) \in L^2_{\mathrm{loc}}(R^3)$. For all $T > 0$, $u(x,t) \in L^\infty([0,T]; L^2_{\mathrm{loc}}(R^3))$ is called a weak solution to (1.1)–(1.2), if

(i) for any vector function $\psi = (\psi_1, \psi_2, \psi_3) \in C_0^\infty(R^3 \times [0,T))$ with $\operatorname{div}\psi = 0$,

$$\int_{R^3} \psi(x,0)u_0(x)\,dx + \int_0^T \int_{R^3} (\psi_t \cdot u + \nabla\psi : u \otimes u)\,dx\,dt = 0; \tag{2.8}$$

(ii) for any scalar function $\phi \in C_0^\infty(R^3 \times [0,T])$,

$$\int_0^T \int_{R^3} \nabla\phi \cdot u\,dx\,dt = 0;$$

(iii)

$$u(x,t) \in Lip([0,T]; H^{-s}_{\mathrm{loc}}(R^3)) \quad \text{for some } s > 0,$$

In (2.8), $u \otimes u$ means a matrix $(u_i u_j)$ and $A : B = \sum_{i,j} a_{ij} b_{ij}$ for two matrixes $A = (a_{ij})$ and $B = (b_{ij})$.

3. Strong convergence

For the approximate solutions $\{u^\varepsilon\}$ stated in Section 2, it follows from Propositions 2.2 that

$$u^{\varepsilon_j} \rightharpoonup u \text{ weakly in } L^2([0,T]; L^2(R^3)) \tag{3.1}$$

for some subsequence $\{u^{\varepsilon_j}\}$ of $\{u^\varepsilon\}$. One of the main concerns is whether such a weak convergence becomes strong. Along this line, Delort gives the following interesting result.

Proposition 3.1 (*Delort [5]*). *Under Assumptions* (A$^+$), *one has*

$$u_1^{\varepsilon_j} u_3^{\varepsilon_j} \rightharpoonup u_1 u_3,$$

$$u_2^{\varepsilon_j} u_3^{\varepsilon_j} \rightharpoonup u_2 u_3,$$

$$\left(u_1^{\varepsilon_j}\right)^2 + \left(u_2^{\varepsilon_j}\right)^2 - \left(u_3^{\varepsilon_j}\right)^2 \rightharpoonup (u_1)^2 + (u_2)^2 - (u_3)^2,$$

in the sense of distributions.

Furthermore, if the weak limit $u(x,t)$ *in* (3.1) *is a solution of* 3-D *axisymmetric Euler equations in the sense of distributions, then* u^{ε_j} *converges strongly to* u *in* $L^2([0,T]; L^2_{\mathrm{loc}}(R^3))$.

Thus, by Proposition 3.1, the key to the existence of classical weak solutions to the one-signed vortex-sheets for 3-D axisymmetric flows lies in whether the weak convergence (3.1) becomes strong. Further investigations on the properties of the approximate solutions are desired. In this section, we will prove that if a sequence of approximate solutions converges strongly in the region away from the symmetry axis, then it has a subsequence which converges strongly in the whole space. We note that this result is true for general initial vorticity.

Our main result is stated as

Theorem 3.2. *For the approximate solutions* $\{u^\varepsilon\}$ *constructed in Proposition* 2.2, *if there exists a subsequence* $\{u^{\varepsilon_j}\} \subset \{u^\varepsilon\}$ *such that for any* $Q \subset\subset R^3 \backslash \{x \in R^3 |\ r=0\}$, *an open set compactly contained in* $R^3\backslash\{x \in R^3 |\ r=0\}$,

$$u^{\varepsilon_j} \longrightarrow u \quad \textit{strongly in} \quad L^2([0,T]; L^2(Q)), \tag{3.2}$$

then there exists a further subsequence of $\{u^{\varepsilon_j}\}$, *still denoted by itself, such that, as* $\varepsilon_j \to 0$,

$$u^{\varepsilon_j} \longrightarrow u \quad \textit{strongly in} \quad L^2([0,T]; L^2_{\mathrm{loc}}(R^3)). \tag{3.3}$$

Proof. The proof is decomposed into the following steps:

Step I: The Integral Equations for the Weak Limit

It follows from assumption (3.2) that there exists a subsequence $\{u^{\varepsilon_j}\} \subset \{u^\varepsilon\}$ such that, for any $Q \subset\subset R^3\backslash\{x \in R^3 | r=0\}$,

$$u^{\varepsilon_j} \longrightarrow u \quad \text{strongly in} \quad L^2([0,T]; L^2(Q)), \ \text{as } \varepsilon_j \to 0, \tag{3.4}$$

where $u = u(x,t)$ is the weak limit of u^ε, satisfying $|u|^2 \in L^1(R^3 \times [0,T])$. We denote the subsequence of u^{ε_j} by itself in the following. Then it is clear that

$$u^{\varepsilon_j} \longrightarrow u \quad a.e.\ (x,t) \in R^3 \times [0,T], \ \text{as } \varepsilon_j \to 0, \tag{3.5}$$

which implies that there are no oscillations in the limit process. On the other hand, using the energy estimate we directly get that $\{|u^{\varepsilon_j}|^2\}$ (its subsequence actually) converges weakly in the sense of measure, that is, as $\varepsilon_j \to 0$,

$$\left(u_i^{\varepsilon_j}\right)^2 dx\,dt \rightharpoonup \mu_i \quad \text{weakly} \quad \text{in} \quad M(R^3 \times [0,T]) \tag{3.6}$$

for $i = 1, 2, 3$, where $M(R^3 \times [0, T])$ is the space of finite Radon measures, $\mu_i \geqslant 0$. By the Lebesgue decomposition and the Radon–Nikodym theorem, there exist $f_i(x, t) \in L^1(R^3 \times [0, T])$ and $\gamma_i \in M(R^3 \times [0, T])$ such that

$$\mu_i = f_i(x, t)\, dx\, dt + \gamma_i, \quad i = 1, 2, 3, \tag{3.7}$$

where $\gamma_i \perp dx\, dt$, i.e., γ_i and $dx dt$ are mutually orthogonal, and γ_i is the singular part of μ_i $(i = 1, 2, 3)$. Thanks to (3.4) and (3.5), one concludes that $f_i = |u_i|^2$ $(i = 1, 2, 3)$ and the support of γ_i, denoted by $Supp\{\gamma_i\}$ $(i = 1, 2, 3)$, is contained in the set $\{(x, t) \in R^3 \times [0, T] \mid r = 0, t \in [0, T]\}$, which will be denoted by $\{r = 0\}$ in the following. In other words, as $\varepsilon_j \to 0$, we have

$$\begin{aligned} \left(u_1^{\varepsilon_j}\right)^2 dx\, dt &\rightharpoonup u_1^2\, dx\, dt + \gamma_1, \\ \left(u_2^{\varepsilon_j}\right)^2 dx\, dt &\rightharpoonup u_2^2\, dx\, dt + \gamma_2, \\ \left(u_3^{\varepsilon_j}\right)^2 dx\, dt &\rightharpoonup u_3^2\, dx\, dt + \gamma_3, \end{aligned} \tag{3.8}$$

weakly in $M(R^3 \times [0, T])$. In (3.8), γ_i $(i = 1, 2, 3)$ are non-negative Radon measures satisfying

$$Supp\{\gamma_i\} \subseteq \{r = 0\}, \quad |\gamma_i| < +\infty, \quad i = 1, 2, 3. \tag{3.9}$$

Here $|\gamma_i|$ means the total variations of γ_i $(i = 1, 2, 3)$.

By construction,

$$\begin{aligned} &\int_{R^3} u_0^{\varepsilon_j}(x)\Phi(x, 0)\, dx + \int_0^T \int_{R^3} (u^{\varepsilon_j}\Phi_t + u^{\varepsilon_j} \otimes u^{\varepsilon_j} : \nabla\Phi)\, dx\, dt \\ &\quad = \int_{R^3} u^{\varepsilon_j}(x, T)\Phi(x, T)\, dx \end{aligned} \tag{3.10}$$

for all $\Phi \in C_0^1(R^3 \times [0, T])$ satisfying $div\, \Phi = 0$.

Note that

$$\begin{aligned} \left| \int_{R^3} u^{\varepsilon_j}(x, T)\Phi(x, T)\, dx \right| &\leqslant \left(\int_{R^3} |u^{\varepsilon_j}|^2\, dx \right)^{1/2} \left(\int_{R^3} |\Phi(x, T)|^2\, dx \right)^{1/2} \\ &\leqslant C \left(\int_{R^3} |\Phi(x, T)|^2\, dx \right)^{1/2}, \end{aligned}$$

where C is a constant independent of ε. This, together with (3.10), implies

$$\begin{aligned}&\int_{R^3} u_0^{\varepsilon_j}(x)\Phi(x,0)\,dx + \int_0^T\int_{R^3}(u^{\varepsilon_j}\Phi_t + u^{\varepsilon_j}\otimes u^{\varepsilon_j} : \nabla\Phi)\,dx\,dt\\ &\quad\leqslant C\left(\int_{R^3}|\Phi(x,T)|^2\,dx\right)^{1/2}.\end{aligned} \tag{3.11}$$

One of our major step (Step III) is to construct the test function $\Phi = (\Phi_1, \Phi_2, \Phi_3) \in C_0^1(R^3 \times [0, T])$ satisfying $div\,\Phi = 0$ and

$$\max_{r\leqslant h, t\in[0,T], x_3\in R} |\partial_i \Phi_k(x,t)| \to 0,\;\; h \to 0 \tag{3.12}$$

for $i, k = 1, 2, 3, \quad i \neq k$.

For such test functions, taking the limit $\varepsilon_j \to 0$ in (3.8) and (3.11), we obtain

$$\begin{aligned}&\left|\int_0^T\int_{R^3}\partial_1\Phi_1\,d\gamma_1 + \int_0^T\int_{R^3}\partial_2\Phi_2\,d\gamma_2 + \int_0^T\int_{R^3}\partial_3\Phi_3\,d\gamma_3\right|\\ &\quad\leqslant\left|\int_{R^3}u_0(x)\Phi(x,0)\,dx\right| + \left|\int_0^T\int_{R^3}(u\Phi_t + u_1^2\partial_1\Phi_1 + u_2^2\partial_2\Phi_2 + u_3^2\partial_3\Phi_3\right.\\ &\qquad+u_1u_2\partial_1\Phi_2 + u_1u_3\partial_1\Phi_3 + u_2u_1\partial_2\Phi_1 + u_2u_3\partial_2\Phi_3 + u_3u_1\partial_3\Phi_1\\ &\qquad\left.+u_3u_2\partial_3\Phi_2)\,dx\,dt\right| + C\left(\int_{R^3}|\Phi(x,T)|^2\,dx\right)^{1/2}.\end{aligned} \tag{3.13}$$

In the limit process above, we have used the following facts:

$$\int_0^T\int_{R^3} u_i^{\varepsilon_j}u_k^{\varepsilon_j}\partial_i\Phi_k\,dx\,dt \longrightarrow \int_0^T\int_{R^3} u_iu_k\partial_i\Phi_k\,dx\,dt,\;\; \varepsilon_j \to 0 \tag{3.14}$$

for $i, k = 1, 2, 3$, and $i \neq k$. Indeed, for any $h > 0$, by assumption (3.2), one has

$$\int_{\{r\geqslant h\}} u_i^{\varepsilon_j}u_k^{\varepsilon_j}\partial_i\Phi_k\,dx\,dt \longrightarrow \int_{\{r\geqslant h\}} u_iu_k\partial_i\Phi_k\,dx\,dt,\quad \varepsilon_j \to 0. \tag{3.15}$$

Here $\{r\geqslant h\} = \{(x,t) \in R^3 \times [0,T]|\; r\geqslant h > 0, t \in [0,T]\}$. It follows from (3.12) that

$$\left|\int_{\{r\leqslant h\}} u_i^{\varepsilon_j}u_k^{\varepsilon_j}\partial_i\Phi_k\,dx\,dt\right| \leqslant C\max_{r\leqslant h, t\in[0,T], x_3\in R}|\partial_i\Phi_k(x,t)| \to 0,\;\; h \to 0 \tag{3.16}$$

for $i \neq k$ $(i, k = 1, 2, 3)$. Here $\{r \leqslant h\} = \{(x, t) \in R^3 \times [0, T] |\ r \leqslant h, t \in [0, T]\}$. Moreover, it is clear that

$$\left| \int_{\{r \leqslant h\}} u_i u_k \partial_i \Phi_k \, dx \, dt \right| \leqslant C \int_{\{r \leqslant h\}} |u|^2 \, dx \, dt \to 0, \ \ h \to 0, \tag{3.17}$$

since $u \in L^2_{\text{loc}}([0, T] \times R^3)$. Thus (3.14) follows from (3.15)–(3.17).

It is clear that in order to prove the theorem, we only need to prove that $\gamma_1 = \gamma_2 = \gamma_3 = 0$.

Step II: $\gamma_1 = \gamma_2 = 0$

Using Proposition 2.3, we can prove directly that $\gamma_1 = \gamma_2 = 0$. Let $\beta(x_3, t) \in C_0^\infty(R \times [0, T])$, and let $\chi = \chi(s)$ be a smooth function satisfying $\chi(s) = 1$ for $|s| \leqslant 1$ and $\chi(s) = 0$ for $|s| \geqslant 2$. We define, for any $\delta > 0$,

$$\varphi_\delta(x, t) = \chi\left(\frac{r}{\delta}\right) \beta(x_3, t)$$

with $r = \left(x_1^2 + x_2^2\right)^{1/2}$. By Proposition 2.3, one has

$$\begin{aligned} &\left| \int_0^T \int_{R^3} \left(\left(u_1^{\varepsilon_j}\right)^2 + \left(u_2^{\varepsilon_j}\right)^2 \right) \varphi_\delta(x, t) \, dx \, dt \right| \\ &\quad \leqslant C\delta^2 \int_0^T \int_{R^3} \frac{1}{1 + x_3^2} \left(\frac{u_r^{\varepsilon_j}}{r} \right)^2 dx \, dt \leqslant C\delta^2, \end{aligned} \tag{3.18}$$

where $C > 0$ is a constant independent of ε_j and δ. Letting $\varepsilon_j \to 0$ in (3.18), in view of (3.8), we get

$$\left| \int_0^T \int_{R^3} (u_1^2 + u_2^2) \varphi_\delta(x, t) \, dx \, dt + \int_0^T \int_{R^3} \beta(x_3, t) \, d(\gamma_1 + \gamma_2) \right| \leqslant C\delta^2, \tag{3.19}$$

where the contant $C > 0$ is same as in (3.18). If $\delta \to 0$, by the dominated convergence theorem, the first term of (3.19) vanishes and we obtain that $\int_0^T \int_{R^3} \beta(x_3, t) \, d(\gamma_1 + \gamma_2) = 0$ for any $\beta(x_3, t) \in C_0^\infty(R \times [0, T])$, hence $\gamma_1 = \gamma_2 = 0$. This implies that the strong convergence of $u_1^{\varepsilon_j}$ and $u_2^{\varepsilon_j}$. It should be noted that at this point, if the initial vorticity were positive, the proof of the theorem would be finished by Proposition 3.1, since it implies that $\gamma_3 = 0$ and $u_3^{\varepsilon_j}$ converges also strongly to u_3. However, under the assumptions of our theorem, there is no sign restriction for the initial vorticity, so we have to prove that $\gamma_3 = 0$.

Step III: $\gamma_3 = 0$

In this section, we will first construct test functions $\Phi = (\Phi_1, \Phi_2, \Phi_3) \in C_0^1(R^3 \times [0, T])$ with $div\, \Phi(x) = 0$. Although it is technical, we will construct the test functions with compact support in R^3, divergence free, and by substituting them into (3.13) we can prove that $\int_0^T \int_{R^3} f(x_3, t)\, d\gamma_3 = 0$ for any $f(x_3, t) \in C_0^\infty(R \times [0, T])$.

For any $R > 0$, let $\Omega \subseteq \{(x_3, t) \in (-\infty, +\infty) \times [0, T] \mid x_3^2 + t^2 < R,\ \ R > 0\}$ be any open set with smooth boundary. We define

$$\begin{aligned}
\Phi_1(x,t) &= -\tfrac{1}{2} x_1 \chi(\tfrac{r}{\delta}) \Big\{ \big[g(x_3,t) + \big(x_3 - x_3^0\big)\, \partial_{x_3} g(x_3,t) \big] \mathbf{I}_\Omega^{\tilde{\delta}} \\
&\qquad + \big(x_3 - x_3^0\big)\, g(x_3,t) \partial_{x_3} \mathbf{I}_\Omega^{\tilde{\delta}} \Big\}, \\
\Phi_2(x,t) &= -\tfrac{1}{2} x_2 \chi(\tfrac{r}{\delta}) \Big\{ \big[g(x_3,t) + \big(x_3 - x_3^0\big)\, \partial_{x_3} g(x_3,t) \big] \mathbf{I}_\Omega^{\tilde{\delta}} \\
&\qquad + \big(x_3 - x_3^0\big)\, g(x_3,t) \partial_{x_3} \mathbf{I}_\Omega^{\tilde{\delta}} \Big\}, \\
\Phi_3(x,t) &= \big[\chi(\tfrac{r}{\delta}) + \tfrac{r}{2\delta} \chi'(\tfrac{r}{\delta}) \big] \big(x_3 - x_3^0\big)\, g(x_3,t) \mathbf{I}_\Omega^{\tilde{\delta}}.
\end{aligned} \tag{3.20}$$

Here $\delta > 0$ is small and $\tilde{\delta} > 0$ is arbitrary. The number $x_3^0 \in (-\infty, +\infty)$ is any fixed real number satisfying $|x_3^0| > R + 3$. The function $\chi = \chi(s)$ is same as in Step II, which is a smooth function satisfying $\chi(s) = 1$ for $|s| \leqslant 1$ and $\chi(s) = 0$ for $|s| \geqslant 2$. And $\mathbf{I}_\Omega^{\tilde{\delta}} = \mathbf{I}_\Omega^{\tilde{\delta}}(x_3, t)$ and $g = g(x_3, t)$ are also smooth functions which will be determined in the sequel.

Regarding $\mathbf{I}_\Omega^{\tilde{\delta}} = \mathbf{I}_\Omega^{\tilde{\delta}}(x_3, t)$: We define the smooth function $\mathbf{I}_\Omega^{\tilde{\delta}} = \mathbf{I}_\Omega^{\tilde{\delta}}(x_3, t)$ satisfying, for any positive number $\tilde{\delta} > R$,

$$\begin{aligned}
&0 \leqslant \mathbf{I}_\Omega^{\tilde{\delta}} \leqslant 1, \\
&\mathbf{I}_\Omega^{\tilde{\delta}}(x_3,t) \equiv 1, \quad (x_3,t) \in \Omega, \\
&\mathbf{I}_\Omega^{\tilde{\delta}}(x_3,t) \equiv 0, \quad |x_3| > \tilde{\delta}, \\
&|\partial_{x_3} \mathbf{I}_\Omega^{\tilde{\delta}}(x_3,t)| \leqslant \frac{C}{\tilde{\delta}},
\end{aligned} \tag{3.21}$$

where C is a constant independent of $\tilde{\delta}$.

Regarding $g = g(x_3, t)$: We define the smooth function $g = g(x_3, t) \in C_0^\infty((-\infty, +\infty) \times [0, T])$ satisfying, for any $f(x_3, t) \in C_0^\infty(\Omega)$,

$$\begin{aligned}
&g + \big(x_3 - x_3^0\big)\, \partial_{x_3} g = f \quad \text{in} \quad \Omega, \\
&g(x_3,t) = 0, \quad |x_3| > R + 2.
\end{aligned} \tag{3.22}$$

Indeed, the function $g(x_3,t)$ can be constructed in the following way. First we solve the equation in (3.22) with $g(0,t)=0$ to get a smooth solution $\tilde{g}(x_3,t)$ as

$$\tilde{g}(x_3,t)=\frac{1}{x_3-x_3^0}\int_0^{x_3} f(y,t)\,dy$$

for $|x_3|\leqslant R+2, t\in[0,T]$, noting that $|x_3^0|>R+3$. Then using the smooth cut-off function $\tilde{\chi}(s)$ satisfying $\tilde{\chi}(s)=1$ for $|s|\leqslant R$ and $\tilde{\chi}(s)=0$ for $|s|\geqslant R+1$, we define the desired function $g(x_3,t)$ as

$$g(x_3,t)=\begin{cases}\tilde{g}(x_3,t)\tilde{\chi}(x_3), & |x_3|\leqslant R+2, t\in[0,T],\\ 0, & |x_3|>R+2, t\in[0,T].\end{cases} \tag{3.23}$$

It can be verified directly that the test functions given by (3.20)–(3.22) satisfy

$$\Phi=(\Phi_1,\Phi_2,\Phi_3)\in C_0^1(R^3\times[0,T]),\quad div\,\Phi=0.$$

And it should be noted that the support of $\{\Phi(x,t)\}$ are included in the set $\{(x,t)\in R^3\times[0,T]\,||r|\leqslant 2\delta, |x_3|\leqslant 2R\}$ and restrictions (3.12) are satisfied. Thus the integral inequality (3.13) holds for the test functions defined by (3.20).

Note that

$$\begin{aligned}\partial_1\Phi_1 &=\left[-\tfrac{1}{2}\chi(\tfrac{r}{\delta})-\tfrac{x_1^2}{2\delta r}\chi'(\tfrac{r}{\delta})\right]\Big\{\left[g(x_3,t)+\left(x_3-x_3^0\right)\partial_{x_3}g(x_3,t)\right]\mathbf{I}_\Omega^{\tilde{\delta}}\\ &\quad+\left(x_3-x_3^0\right)g(x_3,t)\partial_{x_3}\mathbf{I}_\Omega^{\tilde{\delta}}\Big\},\\ \partial_2\Phi_2 &=\left[-\tfrac{1}{2}\chi(\tfrac{r}{\delta})-\tfrac{x_2^2}{2\delta r}\chi'(\tfrac{r}{\delta})\right]\Big\{\left[g(x_3,t)+\left(x_3-x_3^0\right)\partial_{x_3}g(x_3,t)\right]\mathbf{I}_\Omega^{\tilde{\delta}}\\ &\quad+\left(x_3-x_3^0\right)g(x_3,t)\partial_{x_3}\mathbf{I}_\Omega^{\tilde{\delta}}\Big\},\\ \partial_3\Phi_3 &=\left[\chi(\tfrac{r}{\delta})+\tfrac{r}{2\delta}\chi'(\tfrac{r}{\delta})\right]\Big\{\left[g(x_3,t)+\left(x_3-x_3^0\right)\partial_{x_3}g(x_3,t)\right]\mathbf{I}_\Omega^{\tilde{\delta}}\\ &\quad+\left(x_3-x_3^0\right)g(x_3,t)\partial_{x_3}\mathbf{I}_\Omega^{\tilde{\delta}}\Big\}\end{aligned} \tag{3.24}$$

and

$$\partial_3\Phi_3\,|_{r=0,(x_3,t)\in Supp\{\gamma_3\}}=f(x_3,t). \tag{3.25}$$

Substituting these test functions $\Phi = (\Phi_1, \Phi_2, \Phi_3)$ into the integral inequality (3.13), noticing that $\gamma_1 = \gamma_2 = 0$ by Step II, we have

$$\begin{aligned}
&\left|\int_0^T \int_{R^3} f(x_3,t)\,d\gamma_3\right| \\
&\quad\leqslant \left|\int_{R^3} u_0(x)\Phi(x,0)\,dx\right| + \left|\int_0^T \int_{R^3} u\Phi_t\,dx\,dt\right| \\
&\qquad + \left|\int_0^T \int_{R^3} (u_1^2\partial_1\Phi_1 + u_2^2\partial_2\Phi_2 + u_3^2\partial_3\Phi_3)\,dx\,dt\right| \\
&\qquad + \left|\int_0^T \int_{R^3} (u_1u_2\partial_1\Phi_2 + u_2u_1\partial_2\Phi_1 + u_3u_1\partial_3\Phi_1 + u_3u_2\partial_3\Phi_2)\,dx\,dt\right| \\
&\qquad + \left|\int_0^T \int_{R^3} (u_1u_3\partial_1\Phi_3 + u_2u_3\partial_2\Phi_3)\,dx\,dt\right| \\
&\qquad + C\left(\int_{R^3} |\Phi(x,T)|^2\,dx\right)^{1/2} \\
&\quad\equiv I_1 + I_2 + I_3 + I_4 + I_5 + I_6.
\end{aligned} \tag{3.26}$$

We now estimate the terms on the right-hand side of (3.26) one by one. First, noting that for any fixed $\tilde{\delta} > 0$, we have

$$\left|\left[g(x_3,t) + \left(x_3 - x_3^0\right)\partial_{x_3} g(x_3,t)\right]\mathbf{I}_\Omega^{\tilde{\delta}} + \left(x_3 - x_3^0\right) g(x_3,t)\partial_{x_3}\mathbf{I}_\Omega^{\tilde{\delta}}\right| \leqslant C_1,$$

where C_1 is a constant depending on $\tilde{\delta}, \max_{(x_3,t)\in\bar{\Omega}} |f(x_3,t)|$ and $|\Omega|$ (the volume of Ω). And it is direct to get

$$|\Phi_t| + \|\Phi_t\|_{L^2(R^3\times[0,T])} \leqslant C_2$$

with C_2 a constant depending on $\tilde{\delta}, \max_{(x_3,t)\in\bar{\Omega}} |f_t(x_3,t)|$ and $|\Omega|$. Consequently,

$$\begin{aligned}
|I_1| &\leqslant C_1 \int_{\{|r|\leqslant 2\delta, |x_3|\leqslant R+2\}} |u_0(x)|\,dx, \\
|I_2| &\leqslant C_2 \left(\int_0^T \int_{\{|r|\leqslant 2\delta, |x_3|\leqslant R+2\}} |u|^2\,dx\,dt\right)^{1/2}.
\end{aligned} \tag{3.27}$$

Then, the expressions of (3.24) implies that

$$|I_3| \leqslant C_1 \int_0^T \int_{\{|r|\leqslant 2\delta, |x_3|\leqslant R+2\}} |u|^2\,dx\,dt. \tag{3.28}$$

A similar calculations as in (3.24) shows that $|\partial_1\Phi_2|$ and $|\partial_2\Phi_1|$ are bounded by the constant C_1. And it follows from the construction of the test functions (3.20)–(3.22) that

$$\left|\partial_{x_3}\left\{\left[g(x_3,t)+\left(x_3-x_3^0\right)\partial_{x_3}g(x_3,t)\right]\mathbf{I}_\Omega^{\tilde{\delta}}+\left(x_3-x_3^0\right)g(x_3,t)\partial_{x_3}\mathbf{I}_\Omega^{\tilde{\delta}}\right\}\right|\leqslant C_3,$$

where C_3 is a constant which may depend on $\tilde{\delta}$, $\max_{(x_3,t)\in\bar{\Omega}}|f(x_3,t)|$ and $|\Omega|$. Therefore, we have

$$|I_4|\leqslant (C_1+C_3)\int_0^T\int_{\{|r|\leqslant 2\delta,|x_3|\leqslant R+2\}}|u|^2\,dx\,dt. \tag{3.29}$$

To estimate the term I_5, we note that

$$|\partial_1\Phi_3|+|\partial_2\Phi_3|\leqslant\frac{1}{2\delta}\left(3\left|\chi'\left(\frac{r}{\delta}\right)\right|+\left|\chi''\left(\frac{r}{\delta}\right)\right|\right)\left|\left(x_3-x_3^0\right)g(x_3,t)\mathbf{I}_\Omega^{\tilde{\delta}}\right|\leqslant\frac{C_4}{\delta}$$

with C_4 a constant depending on $\max_{(x_3,t)\in\bar{\Omega}}|f(x_3,t)|$ and $|\Omega|$. So $|\partial_1\Phi_3|$ and $|\partial_2\Phi_3|$ are not bounded uniformly on δ. However, similar to the proof in Step II, we use Proposition 2.3 to get some decay rate on u_1 and u_2 near the symmetry axis in L^2-norm. More precisely, by Proposition 2.3, we have

$$\begin{aligned}
&\int_0^T\int_{\{r\leqslant 2\delta,|x_3|\leqslant R+2\}}\left[\left(u_1^\varepsilon\right)^2+\left(u_2^\varepsilon\right)^2\right]dx\,dt\\
&\quad\leqslant 4\delta^2(1+(R+2))^2\int_0^T\int_{\{|r|\leqslant 2\delta,|x_3|\leqslant R+2\}}\frac{1}{1+x_3^2}\left(\frac{u_r^\varepsilon}{r}\right)^2dx\,dt\\
&\quad\leqslant C(T,R)\delta^2\left(\|u_0^\varepsilon\|_{L^2}^2+\left\|\frac{\omega_0^\varepsilon}{r}\right\|_{L^1}\right)\\
&\quad\leqslant C_5\delta^2,
\end{aligned}$$

where C_5 is an absolute constant depending on T, R and $\|u_0\|_{L^2}$, $\|\frac{\omega_0}{r}\|_{L^1}$. So

$$\left|\int_0^T\int_{\{|r|\leqslant 2\delta,|x_3|\leqslant R+2\}}\left[(u_1)^2+(u_2)^2\right]dx\,dt\right|\leqslant C_5\delta^2.$$

Consequently, by Hölder inequality, we have

$$|I_5|\leqslant C_4C_5^{\frac{1}{2}}\int_0^T\int_{\{|r|\leqslant 2\delta,|x_3|\leqslant R+2\}}|u_3|^2\,dx\,dt. \tag{3.30}$$

Finally, it is clear that

$$|I_6| \leqslant C \left(\int_{\{|r| \leqslant 2\delta, |x_3| \leqslant R+2\}} |\Phi(x,T)|^2 \, dx \right)^{1/2} \leqslant C_6 \delta^2, \tag{3.31}$$

where C_6 is a constant depending on T, R and $\max_{(x,t)\in R^3\times[0,T]} |\Phi|$.

Substituting estimates (3.27)–(3.31) into (3.26) leads to

$$\begin{aligned} &\left| \int_0^T \int_{R^3} f(x_3,t)\, d\gamma_3 \right| \\ &\quad \leqslant C_1 \int_{\{|r| \leqslant 2\delta, |x_3| \leqslant R+2\}} |u_0(x)| \, dx \\ &\qquad + \left(C_1 + C_2 + C_3 + C_4 C_5^{\frac{1}{2}} \right) \int_0^T \int_{\{|r| \leqslant 2\delta, |x_3| \leqslant R+2\}} |u|^2 \, dx \, dt + C_6 \delta^2, \end{aligned} \tag{3.32}$$

where C_i $(i = 1, \ldots, 6)$ are constants mentioned above. After letting $\delta \to 0$ in (3.32), we get $\int_0^T \int_{R^3} f(x_3,t)\, d\gamma_3 = 0$ for all $f(x_3,t) \in C_0^\infty(\Omega)$. By the arbitrariness of Ω, one has $\int_0^T \int_{R^3} f(x_3,t)\, d\gamma_3 = 0$ for all $f(x_3,t) \in C_0^\infty(R \times [0,T])$ and $\gamma_3 = 0$.

The proof of the theorem is finished. □

As a direct corollary of Theorem 3.2, we have

Theorem 3.3. *Under the assumptions of Theorem* 3.2, *there exists a global weak solutions for the* 3-D *axisymmetric Euler equations without swirls when the initial data is a vortex-sheets.*

4. Concluding remarks

Theorem 3.2 shows that, when the initial data is a vortex-sheets, the key point to get the global weak solutions for the 3-D axisymmetric Euler equations without swirls is to get the strong convergence of the sequences of the approximate solutions in the region away from the axis. One sufficient condition is obtained recently by using the decay rate of the maximum vorticity function [12], which is, the estimate

$$\max_{t\in[0,T]} \int_{|x-x_0| \leqslant R} |\omega^\varepsilon(x,t)| \, dx \leqslant C(K,\delta) R \left(\ln\left(\frac{1}{R}\right) \right)^{-\beta} \tag{4.1}$$

holds uniformly for $x_0 \in \Omega_\delta^K \equiv \{x \in R^3 |\ r > \delta > 0,\ |x| < K < +\infty\}$. Here $R > 0$ is sufficiently small and $\beta > 1$, and the constant $C(K,\delta)$ depends on K, δ. Meanwhile, for the one-signed initial vortex-sheets data, one has (see [12])

$$\max_{t\in[0,T]} \iiint_{|x-x_0|\leqslant R} |\omega(x,t)|\,dx \leqslant C(K,\delta)R\left(\ln\left(\frac{1}{R}\right)\right)^{-\frac{1}{2}}, \tag{4.2}$$

where x_0, R and $C(K,\delta)$ are same as in (4.1). There is still a gap between (4.1) and (4.2). It is noted that in the region away from the symmetry axis, the 3-D axisymmetric Euler equations without swirls look like the 2-D Euler equations. And the existence of weak solutions of the 2-D Euler equations with one-signed vortex-sheets initial vorticity has been solved. It is expected that the 3-D axisymmetric vortex-sheets problem are solvable.

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